Prop (Clairaut's Theorem): If $f(x,y)$ has continuous mixed second order partial derivatives on an open disk, then $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ on the disk

Notation: $f_x = \frac{\partial f}{\partial x}$ $\quad f_y = \frac{\partial f}{\partial y}$

$$f_{xx} = (f_x)_x = \frac{\partial^2 f}{\partial x^2}$$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}[f]\right] = \frac{\partial^2 f}{\partial y \partial x}$$

*Prefer this notation in proof

Proof: Let f have continuous second order mixed partial derivatives on an open disk $D \subseteq \mathbb{R}^2$ and suppose $(a,b) \in D$

Consider:

$$\Delta(h) := (f(a+h,b+h) - f(a+h,b)) - (f(a,b+h) - f(a,b))$$

Define $\alpha(x) = f(x, b+h) - f(x,b)$ and notice

$$\alpha(a+h) - \alpha(a) = (f(a+h,b+h) - f(a+h,b)) - (f(a,b+h) - f(a,b)) = \Delta h$$

For all $h \neq 0$ where $(a+h,b), (a+h,b+h), (a,b+h) \in D$

By the $\overbrace{\text{mean value theorem}}^{\text{MVT}}$, for every given h there is a c_h with $|a - c_h| \leq |h|$ so that

$$\underbrace{h\alpha'(c_h) = \alpha(a+h) - \alpha(a)}_{\text{comes from MVT}} = h(f_x(c_h, b+h) - f_x(c_h, b))$$

$$\therefore \Delta h = \alpha(a+h) - \alpha(a)$$
$$= h(f_x(c_h, b+h) - f_x(c_h,b)) \quad \star$$

Next apply MVT to $\beta(y) = f_x(c_h, y)$ to obtain a d_h with $|b - d_h| \leq |h|$ so that

$$h\beta'(d_h) = \beta(b+h) - \beta(b) = f_x(c_h, b+h) - f_x(c_h, b)$$

$\therefore$ substituting into $\bigstar$ yields

$$\begin{aligned}\Delta h &= h\left(f_x(c_h, b+h) - f_x(c_h, b)\right)\\ &= h\left(h\beta'(d_h)\right)\\ &= h^2 (f_x)_y (c_h, d_h)\\ &= h^2 f_{xy}(c_h, d_h)\end{aligned}$$

We may repeat this argument to obtain (γ_h, δ_h) for all h s.t.

$$|a - \gamma_h| \leq |h| \qquad |b - \delta_h| \leq |h|$$

$$\Delta h = h^2 f_{yx}(\gamma_h, \delta_h)$$

Notice by construction that:

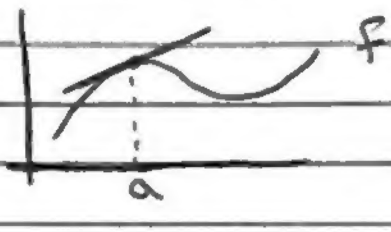
$$\lim_{h\to 0} (c_h, d_h) = (a, b) = \lim_{h \to 0} (\gamma_h, \delta_h)$$

Finally we have:

$$\begin{aligned} f_{xy}(a,b) &= f_{xy}\left(\lim_{h\to 0}(c_h, d_h)\right)\\ &= \lim_{h\to 0} f_{xy}(c_h, d_h) && \leftarrow \text{continuity}\\ &= \lim_{h\to 0} \frac{\Delta h}{h^2} && \leftarrow \text{computed this equality}\\ &= \lim_{h \to 0} f_{yx}(\gamma_h, \delta_h)\\ &= f_{yx}\left(\lim_{h\to 0}(\gamma_h, \delta_h)\right) && \leftarrow \text{continuity}\\ &= f_{yx}(a,b) && \leftarrow \text{proved the result}\end{aligned}$$

14.4 Linear Approximation of Multivariable Functions

Idea: In calc I, near a point on graph (f), f is "approximated well" by the tangent line



as $x \to a$, the error approximating f by the tangent line goes to 0

In calc III, we approximate graph (f) near a point by tangent (hyper)plane instead

for more than 2 variables

In calc I, the tangent line had the formula

$$y - f(a) = f'(a)(x-a)$$

For a function $f(x,y)$, to approximate f near (a,b), we get formula:

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Hence the linear approximation to f at (a,b) is:

$$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

Ex: Find an equation of the tangent plane to $f(x,y) = x^2 + xy - y^2$ at $(1,2)$

Using the formula $z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$

$f_x(x,y) = 2x + y$ $\qquad$ $f_y(x,y) = x - 2y$

$f_x(1,2) = 2\cdot 1 + 2 = 4$ $\qquad$ $f_y(1,2) = 1 - 2\cdot 2 = -3$

$f(1,2) = 1^2 + 1\cdot 2 - 2^2 = -1$

hence the tangent plane is:

$z = 4(x-1) - 3(y-2) - 1$

$$\underline{f(x,y) \approx z = 4(x-1) - 3(y-2) - 1}$$

Ex: Compute the tangent plane to $f(x,y) = \ln(x-2y)$ at $(3,1,0)$ $\qquad$ 12

We need to compute the tangent plane

$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$f(3,1) = \ln(3-2\cdot 1) = 0$

$f_x(x,y) = \frac{1}{x-2y} \cdot 1$ $\qquad$ $f_y(x,y) = \frac{1}{x-2y}(-2)$

$f_x(3,1) = \frac{1}{3-2\cdot 1} \cdot 1 = 1$ $\qquad$ $f_y(3,1) = \frac{1}{3-2\cdot 1}(-2) = -2$

$\therefore \underline{z = 0 + 1(x-3) - 2(y-1)}$

Definition: Let f be a function of variables $x_1, x_2, \ldots, x_n$. The total differential of f is:

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \ldots + \frac{\partial f}{\partial x_n} dx_n$$

Ex: Compute the total differential of $f(x,y,z) = e^x y^2 (z-5)^{1/2}$

$$f_x(x,y,z) = e^x y^2 (z-5)^{1/2}$$
$$f_y(x,y,z) = 2 e^x y (z-5)^{1/2}$$
$$f_z(x,y,z) = \tfrac{1}{2} e^x y^2 (z-5)^{-1/2}$$

$$\underline{df = e^x y^2 (z-5)^{1/2} dx + 2e^x y (z-5)^{1/2} dy + \tfrac{1}{2} e^x y^2 (z-5)^{-1/2} dz}$$

Estimate Δf at $(1,1,6)$ to $(1.5, 1.5, 5.5)$

$$\Delta f \approx df \quad \text{where} \quad dx_i \approx \Delta x_i$$
$$\Delta f = f_x(1,1,6)\Delta x + f_y(1,1,6)\Delta y + f_z(1,1,6)\Delta z$$
$$\Delta f = e(1.5-1) + 2e(1.5-1) + \tfrac{1}{2}e(5.5-6)$$
$$= e\left(\tfrac{1}{2} + 2\cdot\tfrac{1}{2} + \tfrac{1}{2}\cdot -\tfrac{1}{2}\right) = \underline{e\cdot\tfrac{5}{4} = \Delta f}$$